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Ballistic missile trajectory prediction using a state transition matrix

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Abstract

A method for the determination of the trajectory of a ballistic missile over a rotating, spherical Earth given only the launch position and impact point has been developed. The iterative solution presented uses a state transition matrix to correct the initial conditions of the ballistic missile state vector based upon deviations from a desired set of final conditions. A six-degree-of-freedom simulation of a ballistic missile is developed to calculate the resulting trajectory. Given the initial state vector of the ballistic missile, the trajectory is simulated and the state transition matrix propagated along the trajectory to the impact point. The error in the final state vector is calculated and elements of the initial state vector are corrected using the state transition matrix. The process is repeated until the ballistic missile impacts the target location within a predefined miss distance tolerance. The result of this research is an analysis tool which accurately solves for the initial state vector of the ballistic missile.
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1. Introduction

For nearly every type of Anti-Ballistic Missile (ABM) defense method conceived, it is necessary to accurately predict the trajectory of the incoming ballistic missile, which is crucial to coordinate defenses, choose the most viable defense option, and launch at the appropriate time to intercept the incoming missile. An accurate prediction of the trajectory also has analysis applications, including the need to portray the missile state vector throughout flight to assist in the development and evaluation of our country's ballistic missile defense concepts and programs. The purpose of this research is to provide a tool that can be used to determine the trajectory of a ballistic missile, given only the launch position and targeted impact point. In post-mission

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analysis and simulation efforts it is difficult to obtain an accurate representation of the trajectory flown by an identified ballistic missile threat, since only the launch and impact points can be determined with surety. The goal of this research is to provide a numerical approach for the prediction of a representative ballistic missile trajectory. The chosen methodology achieves this by iteratively adjusting elements of the initial state vector that define the missile's orientation, until the missile flies a trajectory that results in an impact within a defined distance from the target location.

One of the earliest studies of trajectory prediction and determination was conducted by Johann Heinrich Lambert (1728–1779). His method was used for spacecraft trajectory prediction in order to solve an orbital boundary value problem. Although this research effort focuses on the prediction of ballistic missile trajectories, Lambert's theorem is a useful application to calculate elliptical parameters of the trajectory to generate an estimate. Using this method, the transfer trajectory between two points can be calculated given the transfer time. The technique is based upon an iterative solution that converges on the semi-major axis of the transfer ellipse and is relevant to the problem considered in this study since it provides a methodology for estimating a trajectory as a solution to a two-point boundary value problem [1].

Isaacson and Vaughan [2] describe a method of estimating and predicting ballistic missile trajectories using a Kalman Filter. A template is developed using missile range and altitude data by modeling the ballistic missile flight over a spherical, non-rotating Earth. The trajectory is used as a baseline from which perturbations of the actual trajectory are calculated. Observations are made throughout the missile trajectory using a pair of geosynchronous satellites. The method is used to determine uncertainties in the missile launch point and missile position during flight.

Shapiro [3] describes three methods used for ballistic missile trajectory estimation and prediction. These estimation schemes are the method of maximum likelihood, the weighted least squares method, and a deterministic method. The method of maximum likelihood computes an estimate of the trajectory parameters based upon a probability density function. The weighted least squares technique computes trajectory parameter estimates that minimize the weighted sum of squares of the measurement errors. The deterministic method actually computes the trajectory by solving the known equations that relate the measurements to the trajectory parameters. The method assumes that no corruption of the radar data has taken place. Each of these methods is used to calculate six parameters that define the orbital parameters describing the ballistic missile trajectory. The ballistic missile is modeled as a point mass in a vacuum with gravity being the only force acting upon it. All measurements of the trajectory are made by a single radar system on a non-rotating Earth, and include range, range-rate, and the azimuth and elevation angles.

Danis [4] developed a method for estimating ballistic missile launch parameters using a single pair of angle measurements taken from two satellites, or utilizing both measurements taken from a single satellite. Using a priori information about the trajectory and launch time, the launch location and missile heading are calculated from the geometric relationship between the missile and satellite position. If sufficient measurements are taken during the missile flight, a Kalman filter can be used to assess measurement errors and refine the launch parameter estimation.

In order to iteratively compute a required trajectory, adjustments to elements of the initial state vector are made. These could be adjustments to the initial position, velocity, or vehicle orientation. The solution to the problem considered in this study requires the initial orientation of the missile. This orientation can be described by using a unit quaternion as presented by Bar-Itzhack [5]. He developed an iterative method to compute the initial quaternion corresponding to the orthogonal transformation matrix relating two Cartesian coordinate systems. Beginning with an arbitrary matrix, a method was developed to converge to a given direction cosine matrix using an implicit self-alignment process. Once the direction cosine matrix was determined, a method was presented to iteratively calculate the initial unit quaternion.

Each of these contributions relies on a priori information, such as a known time of flight, or a set of observation data taken during the missile fly out. For the purposes of this research, only the launch position and targeted impact point are known, in addition to missile airframe and performance data. The principle result of this study is a tool that can be used to predict the trajectory of a ballistic missile by iteratively adjusting elements of the initial state vector until the missile flies a trajectory that results in an impact within a defined tolerance of the target location. A detailed discussion of the equations of motion is presented in the following section.

2. Coordinate systems

The coordinate systems used in missile equations of motion are an Earth-Centered-Inertial (ECI) frame $({}^{E}\widehat{X}, {}^{E}\widehat{Y}, {}^{E}\widehat{Z})$, a North-East-Down (NED) frame $({}^{N}\widehat{X}, {}^{N}\widehat{Y}, {}^{N}\widehat{Z})$, and a Body-Fixed (BODY) frame $({}^{B}\widehat{X}, {}^{B}\widehat{Y}, {}^{B}\widehat{Z})$. Fig. 1 depicts the three coordinate systems and their relationship to each other.

The ECI frame has its origin at the center of the Earth and the ${}^{E}\widehat{X}$ and ${}^{E}\widehat{Y}$ axes lie in the equatorial plane. The ${}^{E}\widehat{X}$ axis points toward the Greenwich meridian and the ${}^{E}\widehat{Z}$ axis points toward the North Pole [6]. The NED frame is related to the ECI frame by geocentric latitude, ϕ, longitude, λ, and altitude above the Earth surface, h. The origin of the NED frame is collocated with the missile launch site. The launch site longitude is measured with respect to the origin of the ECI frame and can therefore be used to compute the Earth rotation angle, assuming a fixed launch site. The ${}^{N}\widehat{X}$, ${}^{N}\widehat{Y}$, and ${}^{N}\widehat{Z}$ axes point in the north, east, and down directions, respectively [6,7].

The BODY frame is related to the NED system by a 3–2–1 rotation sequence using the Euler angles Ψ, Θ, and Φ in that order. The positive ${}^{B}\widehat{X}$ axis is coincident with the missile longitudinal axis and points forward through the nose of the missile. The ${}^{B}\widehat{Y}$ axis is directed out the right side and the ${}^{B}\widehat{Z}$ axis points downward, normal to the ${}^{B}\widehat{X}$ and ${}^{B}\widehat{Y}$ axes [7].

All airframe related calculations, with the exception of the equations of motion which require an inertial frame, are performed in the BODY frame. For a positive angle of attack, α, the resulting normal force is positive, and for a positive angle of sideslip, β, the resulting side force is negative. Axial force is defined as positive along the ${}^{B}\widehat{X}$ direction. A positive pitching moment results in a positive rotation about the ${}^{B}\widehat{Y}$ axis, or nose-up when viewed from the rear of the missile. A positive yawing moment results in a positive rotation about the ${}^{B}\widehat{Z}$

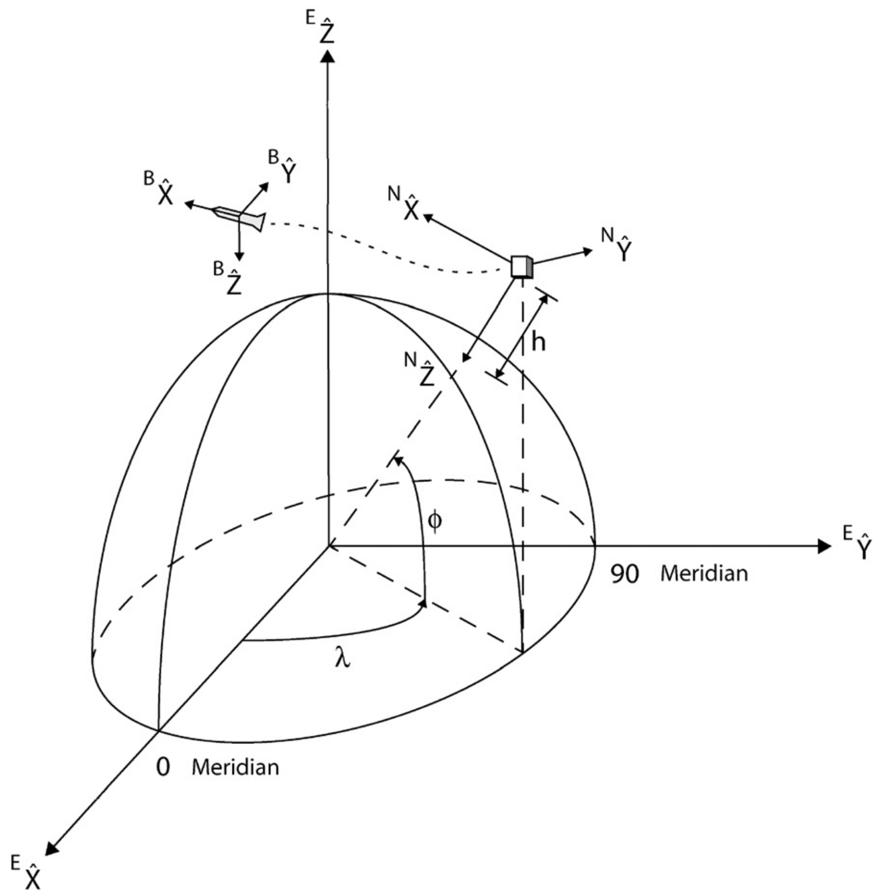

Fig. 1. Coordinate system definition.

axis, or nose-right when viewed from the rear. A positive rolling moment results in a positive rotation about the ${}^{\mathrm{B}}\widehat{X}$ axis.

3. Dynamical model

In developing the equations of motion of the missile, two assumptions are employed: the airframe is assumed to be symmetrical about the longitudinal axis, resulting in aerodynamic coefficient tables that reflect a generic axisymmetric airframe, and; the thrust is assumed to be perfectly aligned with the longitudinal axis of the airframe and is the only propulsive force acting on the missile body.

Using the six-degree-of-freedom (6-DOF) dynamical model described by Etkin [6], the differential equations describing the motion of the ballistic missile are provided in Eqs. (1)–(24). Integration of these equations yields the missile position and velocity state vectors with respect to the NED reference frame

$$
{}^{\mathrm{N}}\ddot{\bar{r}} = {}^{\mathrm{N}}\bar{a}_{\mathrm{TOTAL}} = {}^{\mathrm{N}}\bar{a}_{\mathrm{CG}} + {}^{\mathrm{N}}\bar{a}_{\mathrm{G}} - {}^{\mathrm{N}}a_{\mathrm{COR}} - {}^{\mathrm{N}}\bar{a}_{\mathrm{CEN}} - {}^{\mathrm{N}}\bar{a}_{\mathrm{LS}}, \tag{1}
$$

where

${}^{\mathrm{N}}\bar{a}_{\mathrm{CG}}$ acceleration of the missile airframe center-of-gravity (CG) due to propulsive and aerodynamic forces
${}^{\mathrm{N}}\bar{a}_{\mathrm{G}}$ acceleration due to gravity
${}^{\mathrm{N}}\bar{a}_{\mathrm{COR}}$ Coriolis acceleration
${}^{\mathrm{N}}\bar{a}_{\mathrm{CEN}}$ centripetal acceleration
${}^{\mathrm{N}}\bar{a}_{\mathrm{LS}}$ tangential acceleration of the launch site

The expression depicting the acceleration of the missile airframe CG due to propulsive and aerodynamic forces is

$$
{}^{\mathrm{N}}\bar{a}_{\mathrm{CG}} = {}^{\mathrm{N}}[T]^{\mathrm{B}}\frac{({}^{\mathrm{B}}\overline{F}_{\mathrm{PROP}} + {}^{\mathrm{B}}\overline{F}_{\mathrm{AERO}})}{m}, \tag{2}
$$

where

${}^{\mathrm{B}}\overline{F}_{\mathrm{PROP}}$ propulsion forces acting on the missile airframe
${}^{\mathrm{B}}\overline{F}_{\mathrm{AERO}}$ aerodynamic forces acting on the missile airframe
m missile mass
${}^{\mathrm{N}}[T]^{\mathrm{B}}$ transformation matrix required to rotate from the BODY coordinate system to the NED coordinate system

Since the propulsion forces acting on the airframe are aligned with the longitudinal axis, they do not impart a moment on the airframe. Therefore, the propulsion force acting on the airframe CG along the longitudinal axis is equal to the delivered thrust, and the forces acting along the transverse axes are zero.

The aerodynamic coefficients, C_{A}, C_{Y}, and C_{N} are calculated from table lookups, and are functions of angle of attack, α, angle of sideslip, β, Mach Number, M, and altitude, h. Once the aerodynamic coefficients have been determined, the aerodynamic forces acting on the airframe can be calculated as presented in [8] as

$$
{}^{\mathrm{B}}F_{\mathrm{AERO}_x} = -C_{\mathrm{A}}qS, \tag{3}
$$
$$
{}^{\mathrm{B}}F_{\mathrm{AERO}_{yx}} = -C_{\mathrm{Y}}qS, \tag{4}
$$
$$
{}^{\mathrm{B}}F_{\mathrm{AERO}_z} = -C_{\mathrm{N}}qS, \tag{5}
$$

where

C_{A} aerodynamic coefficient due to axial force
C_{Y} aerodynamic coefficient due to side force
C_{N} aerodynamic coefficient due to normal force
q dynamic pressure
S missile airframe reference area [9]

The boost stage is modeled as a 3-DOF in which aerodynamic forces on the airframe are computed using a ballistic coefficient. The ballistic coefficient, β_c, is the ratio of airframe mass to the zero lift drag coefficient, C_{D0}, and the reference area as

$$\beta_C = \frac{m}{C_{D0}S}. \tag{6}$$

The acceleration acting on the airframe CG as a result of the aerodynamic force during the boost phase is calculated using

$$^{B}a_{CG_x} = -\cos(\alpha)\cos(\beta_C)\left(\frac{q}{\beta_C}\right), \tag{7}$$

$$^{B}a_{CG_y} = -\cos(\alpha)\sin(\beta_C)\left(\frac{q}{\beta_C}\right), \tag{8}$$

$$^{B}a_{CG_z} = -\sin(\alpha)\left(\frac{q}{\beta_C}\right). \tag{9}$$

The gravitational acceleration vector is calculated in the ECI reference frame using the expression

$$^{E}\bar{a}_G = \frac{-\mu_G {}^{E}\bar{r}}{\|{}^{E}\bar{r}\|^3}, \tag{10}$$

where μ_G is the gravitational constant of the Earth and $^{E}\bar{r}$ is the position of the airframe with respect to the ECI frame [1]. The gravitational acceleration vector is transformed to the NED frame using the $^{N}[T]^{E}$ transformation matrix.

The Coriolis and centripetal accelerations due to Earth rotation are calculated using the expressions

$$^{N}\bar{a}_{COR} = 2({}^{N}\bar{\omega}_{LS} \times {}^{N}\dot{\bar{r}}), \tag{11}$$

$$^{N}\bar{a}_{CEN} = {}^{N}\bar{\omega}_{LS} \times ({}^{N}\bar{\omega}_{LS} \times {}^{N}\bar{r}), \tag{12}$$

where
$^{N}\bar{\omega}_{LS}$ the angular velocity of the Earth, which is assumed to be constant
$^{N}\bar{r}$ position vector of the airframe with respect to the NED reference frame
$^{N}\dot{\bar{r}}$ velocity vector of the airframe with respect to the NED reference frame

The tangential acceleration of the launch site is calculated using the expression

$$^{N}\bar{a}_{LS} = {}^{N}[T]^{E}({}^{E}\bar{\omega}_{LS} \times ({}^{E}\bar{\omega}_{LS} \times {}^{E}\bar{r}_{LS})), \tag{13}$$

where
$^{E}\bar{\omega}_{LS}$ the angular velocity of the Earth with respect to the ECI frame
$^{E}\bar{r}_{LS}$ the position of the launch site with respect to the ECI frame

The missile angular acceleration in the BODY frame, can be expressed as

$$^{B}\bar{\alpha} = \frac{{}^{B}\overline{M}_{CG} - ({}^{B}\bar{\omega} \times I {}^{B}\bar{\omega})}{I}, \tag{14}$$

where
$^{B}M_{CG}$ total moment acting about the airframe
$^{B}\bar{\omega}$ missile angular velocity in the BODY frame having components $^{B}\omega_x$, $^{B}\omega_y$, and $^{B}\omega_z$
I airframe moment of inertia in the BODY frame

Integration of Eq. (14) results in the missile angular velocity in the BODY frame. Since it is assumed that the airframe CG location coincides with the longitudinal axis and there are no propulsive forces acting along the transverse axes, the moment about the CG due to propulsive forces is zero. The aerodynamic moment vector is calculated as

$$^{\mathrm{B}}M_{\mathrm{AERO}_x} = \left(C_l + C_{lp}{}^{\mathrm{B}}\omega_x \frac{d}{2v}\right) qSd, \tag{15}$$

$$^{\mathrm{B}}M_{\mathrm{AERO}_y} = \left(C_m + C_{mq}{}^{\mathrm{B}}\omega_y \frac{d}{2v}\right) qSd, \tag{16}$$

$$^{\mathrm{B}}M_{\mathrm{AERO}_z} = \left(C_n + C_{nr}{}^{\mathrm{B}}\omega_z \frac{d}{2v}\right) qSd, \tag{17}$$

where

C_l aerodynamic coefficient due to roll moment
C_m aerodynamic coefficient due to pitch moment
C_n aerodynamic coefficient due to yaw moment
C_{lp} aerodynamic coefficient due to roll damping
C_{mq} aerodynamic coefficient due to pitch damping
C_{nr} aerodynamic coefficient due to yaw damping
v magnitude of the total airframe velocity
d aerodynamic reference length [8]

The total aerodynamic moment acting on the airframe CG is expressed as

$$^{\mathrm{B}}\overline{M}_{\mathrm{AEROTOTAL}} = {}^{\mathrm{B}}\overline{M}_{\mathrm{AERO}} + \overline{L} \times {}^{\mathrm{B}}\overline{F}_{\mathrm{AERO}}, \tag{18}$$

where

$$\overline{L} = \overline{L}_{\mathrm{REF}} - \overline{L}_{\mathrm{CG}}. \tag{19}$$

The position vector of the aerodynamic reference point and CG are $\overline{L}_{\mathrm{REF}}$ and $\overline{L}_{\mathrm{CG}}$, respectively.

Orientation of the airframe is calculated using a unit quaternion of the form

$$\bar{q} = q_4 + q_1\bar{i} + q_2\bar{j} + q_3\bar{k}. \tag{20}$$

The missile quaternion derivatives as a function of the missile angular velocities are provided as

$$\dot{q}_1 = \frac{1}{2}({}^{\mathrm{B}}\omega_x q_4 - {}^{\mathrm{B}}\omega_y q_3 + {}^{\mathrm{B}}\omega_z q_2), \tag{21}$$

$$\dot{q}_2 = \frac{1}{2}({}^{\mathrm{B}}\omega_x q_3 + {}^{\mathrm{B}}\omega_y q_4 - {}^{\mathrm{B}}\omega_z q_1), \tag{22}$$

$$\dot{q}_3 = \frac{1}{2}(-{}^{\mathrm{B}}\omega_x q_2 + {}^{\mathrm{B}}\omega_y q_1 + {}^{\mathrm{B}}\omega_z q_4), \tag{23}$$

$$\dot{q}_4 = \frac{1}{2}(-{}^{\mathrm{B}}\omega_x q_1 - {}^{\mathrm{B}}\omega_y q_2 - {}^{\mathrm{B}}\omega_z q_3). \tag{24}$$

Integration of Eqs. (18)–(24) yields the state vector elements corresponding to the missile quaternion. The NED to BODY transformation matrix,$^{B}[T]^{N}$, is computed from the missile quaternion using

$$^{\mathrm{B}}[T]^{\mathrm{N}} = \begin{bmatrix} q_4^2 + q_1^2 - q_2^2 - q_3^2 & 2(q_1q_2 + q_3q_4) & 2(q_1q_3 - q_2q_4) \\ 2(q_1q_2 - q_3q_4) & q_4^2 - q_1^2 + q_2^2 - q_3^2 & 2(q_2q_3 + q_1q_4) \\ 2(q_1q_3 + q_2q_4) & 2(q_2q_3 - q_1q_4) & q_4^2 - q_1^2 - q_2^2 + q_3^2 \end{bmatrix}. \tag{25}$$

The transformation matrix is composed of successive rotations through the Euler angles Φ, Θ, and Ψ as described in [7].

4. State transition matrix

The missile airframe state vector is comprised of 13 elements that define the missile position, velocity, angular velocity, and quaternion. This state vector is written as

$$\overline{X}(t) = [{}^{\mathrm{N}}x \quad {}^{\mathrm{N}}y \quad {}^{\mathrm{N}}z \quad {}^{\mathrm{N}}\dot{x} \quad {}^{\mathrm{N}}\dot{y} \quad {}^{\mathrm{N}}\dot{z} \quad {}^{\mathrm{B}}\omega_x \quad {}^{\mathrm{B}}\omega_y \quad {}^{\mathrm{B}}\omega_z \quad q_1 \quad q_2 \quad q_3 \quad q_4]^{\mathrm{T}}. \tag{26}$$

The equations of motion are linearized about a reference trajectory, $\overline{X}^*(t)$, resulting in the differential equations for the deviation from, or correction to, the reference trajectory given by

$$\dot{\bar{x}}(t) = A(t)\bar{x}(t), \tag{27}$$

where

$$\bar{x}(t) = \bar{X}(t) - \bar{X}^*(t). \tag{28}$$

The state sensitivity matrix $A(t)$ in Eq. (27) is a (13×13) matrix and contains the first order terms in the Taylor Series expansion of the equations of motion about the reference trajectory [10].

The state transition matrix (STM), $\Phi(t, t_k)$, is a linear transformation which maps the state correction $\bar{x}_k$ at time t_k, into the state correction $\bar{x}(t)$ at time t, as shown by the expression

$$\bar{x}(t) = \Phi(t, t_k)\bar{x}_k. \tag{29}$$

The STM is calculated by integrating the differential equation for $\Phi(t, t_k)$, given by

$$\dot{\Phi}(t, t_k) = A(t)\Phi(t, t_k), \quad \Phi(t_k, t_k) = I. \tag{30}$$

The STM is a fundamental matrix that satisfies the properties [10] of

$$\Phi(t_k, t_k) = I, \tag{31}$$

$$\Phi(t_2, t_0) = \Phi(t_2, t_1)\Phi(t_1, t_0), \tag{32}$$

$$\Phi(t_2, t_1) = \Phi^{-1}(t_1, t_2). \tag{33}$$

Applying these properties of the STM allows Eq. (29) to be rewritten as

$$\begin{aligned}&[\delta^{\mathrm{N}}x \quad \delta^{\mathrm{N}}y \quad \delta^{\mathrm{N}}z \quad \delta^{\mathrm{N}}\dot{x} \quad \delta^{\mathrm{N}}\dot{y} \quad \delta^{\mathrm{N}}\dot{z} \quad \delta^{\mathrm{B}}\omega_x \quad \delta^{\mathrm{B}}\omega_y \quad \delta^{\mathrm{B}}\omega_z \quad \delta q_1 \quad \delta q_2 \quad \delta q_3 \quad \delta q_4]_{\mathrm{f}}^{\mathrm{T}} \\ &\quad = \Phi(t_{\mathrm{f}}, t_0)\,[\delta^{\mathrm{N}}x \quad \delta^{\mathrm{N}}y \quad \delta^{\mathrm{N}}z \quad \delta^{\mathrm{N}}\dot{x} \quad \delta^{\mathrm{N}}\dot{y} \quad \delta^{\mathrm{N}}\dot{z} \quad \delta^{\mathrm{B}}\omega_x \quad \delta^{\mathrm{B}}\omega_y \quad \delta^{\mathrm{B}}\omega_z \quad \delta q_1 \quad \delta q_2 \quad \delta q_3 \quad \delta q_4]_0^T\end{aligned} \tag{34}$$

This dynamical model is then integrated to determine the state vector and the associated STM at any time along the trajectory.

The initial missile position coincides with the origin of the NED coordinate system and must remain fixed throughout the iterative process. The missile airframe is initially at rest, therefore the only elements of the initial state vector that can be used in the iteration are components of the missile quaternion.

Ordinarily, the entire STM would be inverted to solve for corrections to the initial state given deviations in the final state. However, the only known information is the desired impact point location at the final time, t_{f}. Therefore the only known elements of the final state vector corresponding to the true trajectory are the position elements $[x \quad y \quad z]^{\mathrm{T}}$ and the only known deviations from the true trajectory at time t_{f} are $[\delta x_{\mathrm{f}} \quad \delta y_{\mathrm{f}} \quad \delta z_{\mathrm{f}}]^{\mathrm{T}}$. Corrections to the initial state vector that are available for use are changes to the initial missile quaternion, $[\delta q_1 \quad \delta q_2 \quad \delta q_3 \quad \delta q_4]^{\mathrm{T}}$. This simplifies the problem and reduces the number of equations to be solved to three, which are given by

$$\delta x_{\mathrm{f}} = \Phi(1,10)\delta q_1 + \Phi(1,11)\delta q_2 + \Phi(1,12)\delta q_3 + \Phi(1,13)\delta q_4, \tag{35}$$

$$\delta y_{\mathrm{f}} = \Phi(2,10)\delta q_1 + \Phi(2,11)\delta q_2 + \Phi(2,12)\delta q_3 + \Phi(2,13)\delta q_4, \tag{36}$$

$$\delta z_{\mathrm{f}} = \Phi(3,10)\delta q_1 + \Phi(3,11)\delta q_2 + \Phi(3,12)\delta q_3 + \Phi(3,13)\delta q_4. \tag{37}$$

The result is a system of three equations having four unknowns. A minimum norm solution can subsequently be used to calculate an estimate to the initial quaternion. The corrections, $[\delta q_1 \quad \delta q_2 \quad \delta q_3 \quad \delta q_4]^{\mathrm{T}}$, are added to the initial quaternion obtained from the previous iteration, and the process is repeated until the ballistic missile intercepts the target location within a predetermined tolerance.

The minimum norm solution is given [11] as

$$x_k = H^{\mathrm{T}}(HH^{\mathrm{T}})^{-1}y, \tag{38}$$

where the mapping matrix H consists of the elements of the STM shown in Eqs. (35)–(37), and the observation matrix, y, is given as

$$y = [\,\delta x \quad \delta y \quad \delta z]^{\mathrm{T}} \tag{39}$$

Eq. (38) is solved for the sub-matrix of state corrections, $\bar{x}_k$, corresponding to corrections to the quaternions $[\,\delta q_1 \quad \delta q_2 \quad \delta q_3 \quad \delta q_4\,]^{\mathrm{T}}$.

5. Simulation description

The primary limitation of the simulation developed in this study involves the method in which the boost phase is modeled. Typically, there is a complex, threat-specific algorithm for determining the boost phase of the trajectory. Some ballistic missile systems fly a pre-programmed boost trajectory profile and adjust the burnout time in order to achieve a targeted downrange distance. Other systems, such as those using solid fuel as a propellant, adjust the trajectory during boost and keep the burnout time constant. Some systems use a combination of both approaches, where the boost trajectory and burnout time are both adjusted to achieve a desired downrange distance. To more accurately model this portion of the trajectory, a threat-specific guidance computer and autopilot must be simulated. While improving the fidelity of the simulation tool on a case-by-case basis, it will not interfere with the iterative algorithm that is the focus of this study.

The initial airframe state vector consists of position, which must remain unchanged, velocity and angular velocity, which are initially set equal to zero, and the missile quaternion depicting the initial airframe orientation. Therefore, the only components of the initial state vector that can be adjusted are the elements making up the missile quaternion. This presents a dilemma since most ballistic missiles are vertically launched before a pre-programmed boost trajectory profile or adjustable boost trajectory profile reorients the airframe to intercept the target.

To provide a generic simulation that can be applied within the scope of this research, certain simplifying assumptions must be made. The boost trajectory is modeled as a 3-DOF segment in which the missile is launched at an adjustable elevation and azimuth angle and the booster burnout time is held constant. To maximize simulation efficiency, only two stages are modeled, boost and coast, here the coast stage consists of the flight of the re-entry vehicle (RV). The RV is assumed to have no propulsive capability or staging, therefore its mass is constant.

The last assumption is that the Earth is rotating and spherical. This simplifies the calculations for launch position and impact point as well as the equations of motion calculations since $J2$, i.e., oblateness, perturbation effects are not included in the model.

The following section is a detailed description of the ballistic missile trajectory prediction simulation, and a flow chart depicting the algorithm is shown in Fig. 2. The simulation begins by reading the appropriate data files and obtaining the threat specific information. The launch site and target positions are converted from latitude and longitude to the ECI frame and an estimate of the launcher elevation and azimuth angles required to intercept the target is calculated. This is followed by several initialization steps in which the state vector, state sensitivity matrix, state transition matrix, and other necessary values are initialized.

Once initialization is complete, the mass properties and thrust are calculated from table lookups and the velocity of the airframe with respect to the wind, ${}^{\mathrm{B}}\overline{V}_{\mathrm{WIND}}$, is calculated. The thrust data is used to calculate the propulsion force and moment acting about the airframe CG. The angular velocity of the airframe and ${}^{\mathrm{B}}\overline{V}_{\mathrm{WIND}}$ are then used to calculate the aerodynamic force and moment acting about the airframe CG. The propulsion and aerodynamic forces are summed and used to calculate the resulting acceleration, ${}^{\mathrm{B}}\bar{a}_{\mathrm{CG}}$. The propulsion and aerodynamic moments are summed to produce the total moment acting about the airframe CG due to propulsion and aerodynamic forces, ${}^{\mathrm{B}}\overline{M}_{\mathrm{CG}}$.

The state sensitivity matrix, $A(t)$, is calculated with the current state vector and the equations of motion and STM are integrated using a Runge–Kutta numerical integrator. Once the state has been updated and the current STM calculated, the algorithm returns and checks for missile impact. If the missile has not impacted the Earth, the process is repeated until the complete trajectory is obtained. Once the trajectory is calculated, the miss distance between the missile impact point and the target location is determined, taking into account target movement due to Earth rotation. If the miss distance is less than a user defined tolerance, the iterative process is complete and execution of the program is terminated, otherwise corrections to the launcher elevation and azimuth angles are required.

Following a failed tolerance condition, the portion of the STM used in the solution of Eq. (34) is accumulated and the system of equations is solved using the minimum norm solution technique. The correction to the airframe initial quaternion is obtained and summed with the previous values of the quaternion. The algorithm returns to the initialization steps, re-initializes the state vector, STM, and state sensitivity matrix, and repeats the process until convergence.

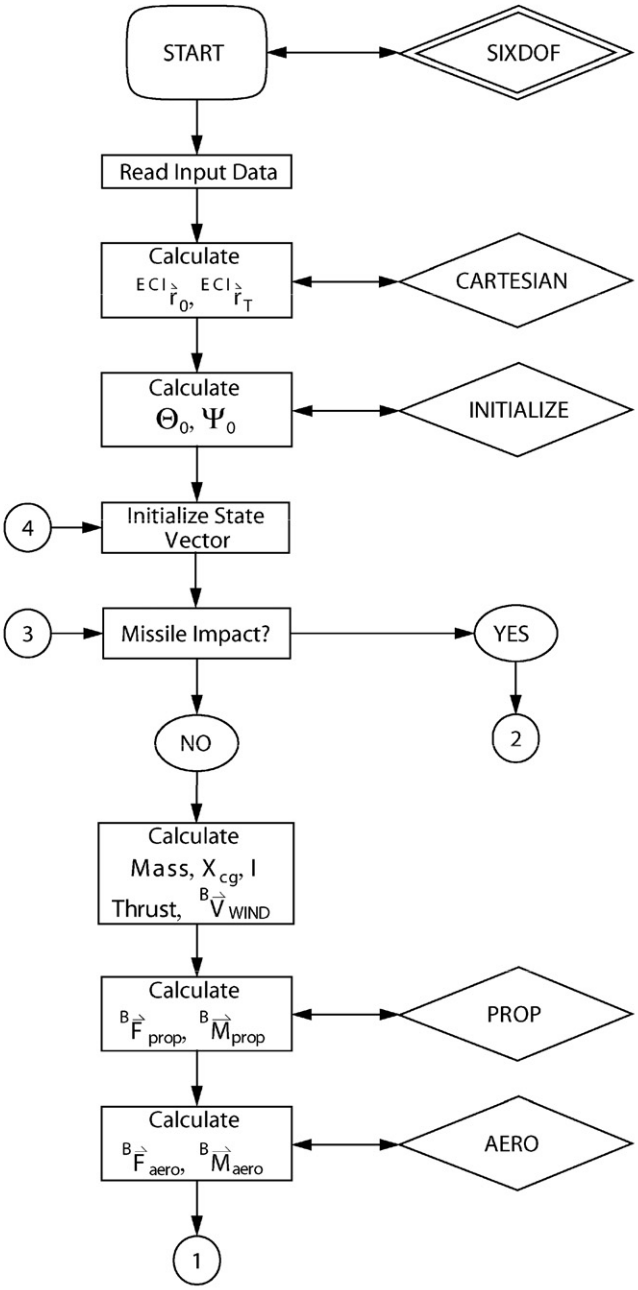

Fig. 2. Simulation order of operation.

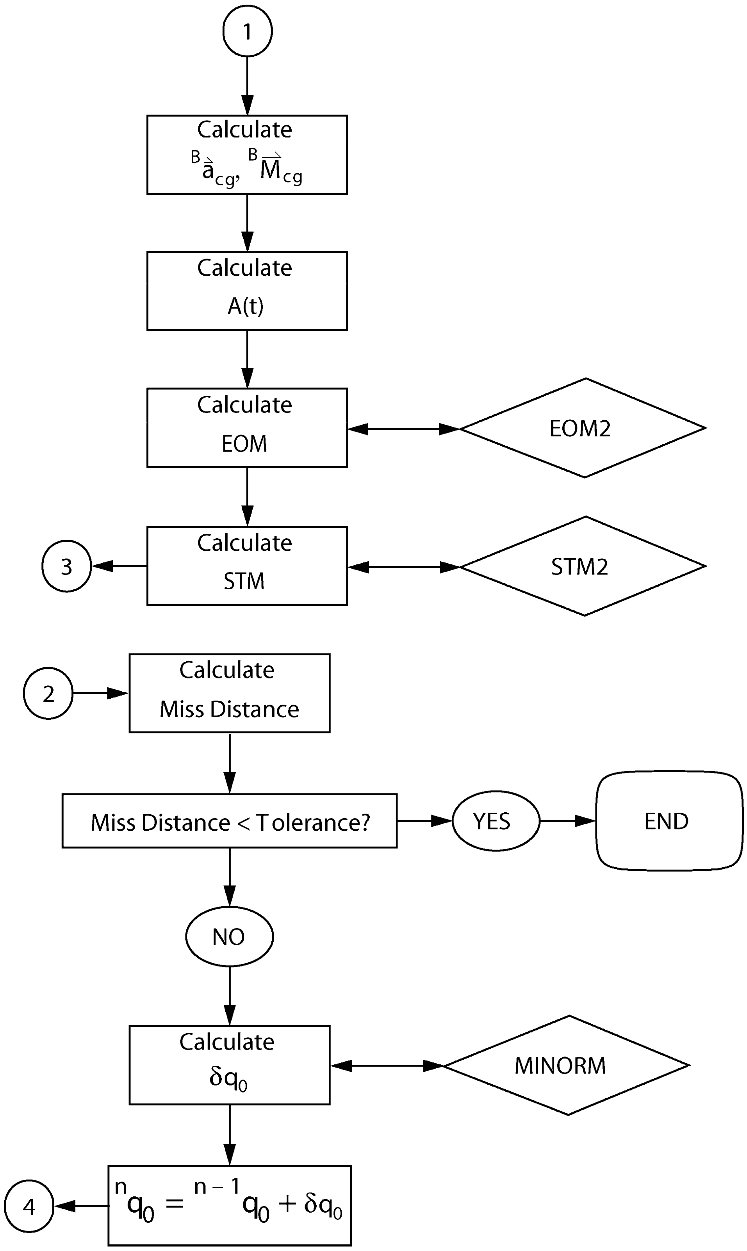

Fig 2. (*continued*)

Within the simulation, Cartesian coordinates are calculated from spherical coordinates using the relationships [12]

$$ {}^{ECI}r_x = (r_E + h)\cos(\lambda)\cos(\phi), \tag{40}$$

$$ {}^{ECI}r_y = (r_E + h)\sin(\lambda)\cos(\phi), \tag{41}$$

$$ {}^{ECI}r_z = (r_E + h)\sin(\phi). \tag{42}$$

A polynomial curve fit is used to estimate the required initial launcher elevation angle. The range angle, Λ, must be calculated to compute an estimate of the required launcher azimuth angle to intercept the target. The range angle is a sum of the powered flight range angle, free flight range angle, and the re-entry range angle [13]. Λ is calculated from the launch site and target latitude and longitude using the expression

$$\cos(\Lambda) = \sin(\phi_0)\sin(\phi_f) + \cos(\phi_0)\cos(\phi_f)\cos(\Delta\lambda), \tag{43}$$

where the difference in longitude between the launch point and the target is denoted by $\Delta\lambda$. The initial launcher azimuth angle estimate is obtained from

$$\cos(\Psi_0) = \frac{\sin(\phi_f) - \sin(\phi_0)\cos(\Lambda)}{\cos(\phi_0)\sin(\Lambda)}. \tag{44}$$

6. Results

A model of the ballistic missile airframe was developed for the simulation. Due to limitations in obtaining airframe, guidance, and autopilot data for the first stage of the missile, the boost phase of the trajectory is modeled as a 3-DOF system. However, complete airframe data sets are available for the remainder of the trajectory for which the missile is modeled as a 6-DOF system.

In order to simulate the trajectory, the ballistic missile was launched at loft angles to produce maximum and minimum range trajectories. Since the booster burnout times are constant for both cases and there is no pitch profile flown during the boost phase, it is expected that the time of flight for the minimum range case is greater than that for the maximum range case. To generate a minimum range trajectory, the ballistic missile is launched at an attitude corresponding to a lofted trajectory. The results from the maximum range case are shown in Figs. 3 and 4. This is a scenario in which the ballistic missile will reach altitudes that are nearly exo-atmospheric. The velocity profile is depicted in Fig. 4. Booster burnout is evident at 72 s, at which point the RV separates and slowly decelerates until apogee is reached at 174 s. As the vehicle accelerates toward Earth through the thin atmosphere the velocity increases until the effects of atmospheric drag are substantial enough to cause the RV to rapidly decelerate, beginning at 280 s into the flight.

Results from the minimum range case are shown in Figs. 5 and 6. As expected, the time of flight is greater due to the higher loft angle at launch. It is apparent from Fig. 5, that the ballistic missile is exo-atmospheric for a large portion of the trajectory. The high eccentricity of the trajectory resulting from the near vertical launch elevation angle is apparent from the velocity profile of Fig. 6. As the RV reaches apogee, the speed decreases dramatically to 175 m/s, then increases as the vehicle accelerates toward Earth under exo-atmospheric conditions. The ballistic missile attains a speed of 1478 m/s at the end of boost for the maximum range case, whereas the speed attained at the end of boost for the minimum range case is 1400 m/s.

A primary intent of this research was to produce a tool to accurately predict the trajectory of a ballistic missile, given only the launch position and impact point. An STM was used to calculate corrections to the initial state vector, given deviations in the final state vector. To test the methodology, six simulated trajectories

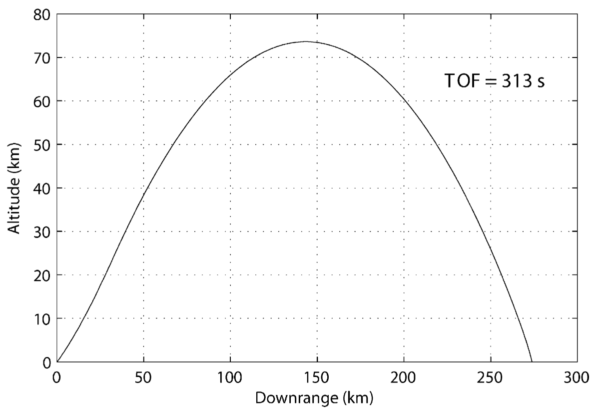

Fig. 3. Altitude vs. downrange for maximum range case.

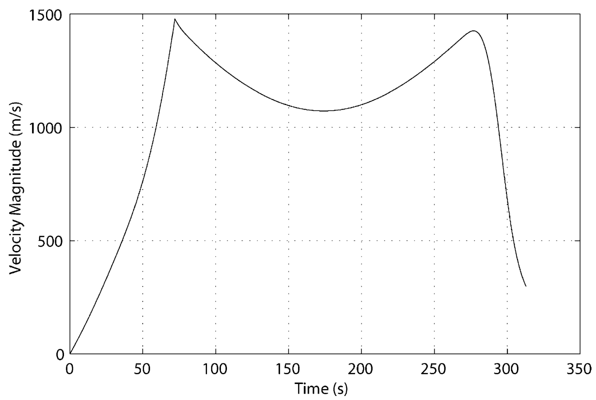

Fig. 4. Velocity profile for maximum range case.

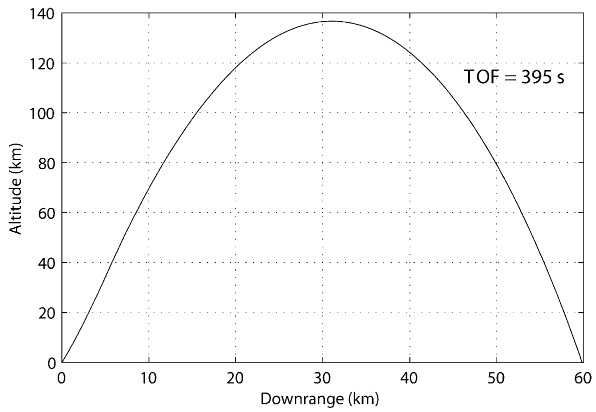

Fig. 5. Altitude vs. downrange for minimum range case.

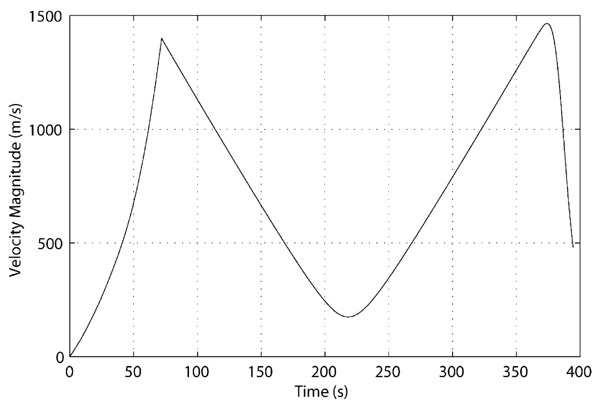

Fig. 6. Velocity profile for minimum range case.

were studied having the initial conditions shown in Table 1. The miss distance tolerance used for these tests was 1500 m. This value was chosen since it is comparable to the accuracy of the ballistic missile systems studied using this simulation.

During initial testing, a typical scenario required 10–15 iterations before a solution within the given tolerance was calculated. To increase the efficiency of the iterative process, two methods were used. The first was an

Table 1
STM test cases

	Case 1	Case 2	Case 3	Case 4	Case 5	Case 6
Launch latitude	89.0	0.0	0.0	0.5	0.0	0.0
Launch longitude	90.0	0.0	0.0	180.5	0.0	0.0
Impact latitude	89.0	1.5	−1.5	−0.5	1.0	1.0
Impact longitude	270.0	0.0	0.0	179.5	1.0	−1.0

improvement on the method of initializing the launcher elevation angle for the first iterative pass. The elevation angle required to impact the target location is a function of missile performance capabilities and target downrange distance. To provide a priori information, five fly outs with varying launcher elevation angles were executed and achieved downrange distance recorded. A plot of launcher elevation angle versus downrange distance was developed and a third order polynomial curve fit to this plot was calculated, as shown in Fig. 7. The polynomial equation that relates launcher elevation angle and achieved downrange distance is

$$\theta_0 = -3.2624 \times 10^{-15} r^3 + 1.2255 \times 10^{-9} r^2 - 2.2591 \times 10^{-4} r + 94.6494. \tag{45}$$

Using this equation, the initial launcher elevation angle is calculated as a function of target downrange distance. This method is not intended to give an accurate value for the required launcher elevation angle to intercept the target, but to provide an estimate that is sufficient to provide a convergent solution.

The second method employed was a finer adjustment to the iterative process. It involved weighting the corrections calculated using the STM with a scale factor based upon missile–target miss distance. The equation used is

$$\mathrm{SF} = \frac{\Delta}{\mathrm{TOL} \times 100} + 1.0, \tag{46}$$

where

Δ missile–target miss distance
TOL user defined tolerance

Each correction, $[\,\delta q_1 \quad \delta q_2 \quad \delta q_3 \quad \delta q_4\,]^{\mathrm{T}}$, is multiplied by the scale factor and then added to the initial quaternion obtained from the previous iteration.

Using these two methods to improve solutions convergence resulted in a substantial reduction in the number of iterations required.

Results of the first test case are shown in Table 2. These results show the robustness of the STM calculations. The scenario is an "over the pole" trajectory. It was expected that corrections to adjust for downrange

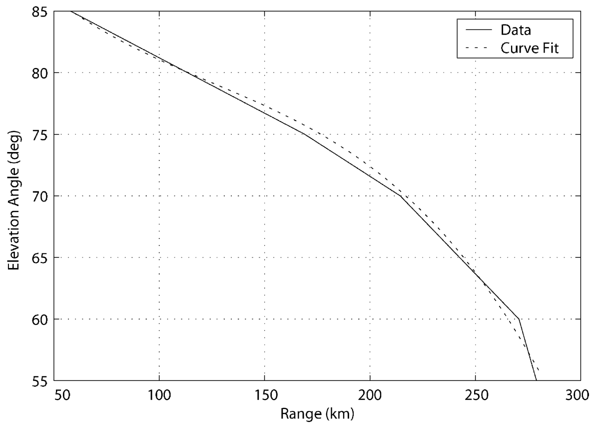

Fig. 7. Polynomial curve fit comparison.

Table 2
STM Case 1 test results

Iteration	Miss distance (m)	Launcher elevation angle (in degrees)	Launcher azimuth angle (in degrees)
1	5348.02	69.10	0.03
2	1802.52	68.94	−1.05
3	1085.40	68.84	−1.12

miss distance would be required in the launcher elevation plane, and adjustments to account for the Earth's rotation would be necessary for the launcher azimuth plane.

The second case examines the effects of Earth rotation on the required launch angles and verifies symmetry in the corrections calculated using the STM. As shown in Table 3 the adjustments made in the launcher azimuth plane result in a launch direction to the Northeast. This is expected since the target is located due North of the launch site, thus a launch towards the Northeast is required to intercept a target located on the rotating Earth.

Symmetry in the launch calculations is verified by comparing the results of Case 2 with those obtained for Case 3. It was expected that the results for Case 3, in which the target is located due South, would be identical to those obtained for Case 2, except for the launcher azimuth corrections. As shown in Table 4, the launcher elevation angles and miss distance remain the same. The launcher azimuth angles differ by 180°, reflecting the change in target location.

The fourth test case studied was used to test the STM correction calculations for a scenario in which the missile passed over the equator and 180° meridian. The results for this case are presented in Table 5. As expected, the solution converged to a value within the defined tolerance without encountering numerical discontinuities in the calculations.

The last test cases studied were used to demonstrate the effects caused by the launcher elevation angle estimation method using the polynomial curve fit. The target is located at an azimuth of 45° to the Northeast for the first case, and 45° to the Northwest for the second case. The curve fit was generated by flying five trajectories, in a due West direction. Therefore it was expected that the launcher elevation angle estimate would be

Table 3
STM Case 2 test results

Iteration	Miss distance (m)	Launcher elevation angle (in degrees)	Launcher azimuth angle (degrees)
1	6463.88	75.91	0.01
2	2619.72	75.55	0.68
3	1280.79	75.42	0.68

Table 4
STM Case 3 test results

Iteration	Miss distance (m)	Launcher elevation angle (in degrees)	Launcher azimuth angle (in degrees)
1	6463.88	75.91	179.99
2	2619.72	75.55	179.32
3	1280.79	75.42	179.32

Table 5
STM Case 4 test results

Iteration	Miss distance (m)	Launcher elevation angle (in degrees)	Launcher azimuth angle (in degrees)
1	4422.42	76.73	−135.00
2	2028.97	76.52	−135.51
3	1035.97	76.42	−135.52

Table 6
STM Case 5 test results

Iteration	Miss distance (m)	Launcher elevation angle (in degrees)	Launcher azimuth angle (in degrees)
1	5911.29	76.73	45.00
2	2556.99	76.42	45.50
3	1207.33	76.29	45.50

Table 7
STM Case 6 test results

Iteration	Miss distance (m)	Launcher elevation angle (in degrees)	Launcher azimuth angle (in degrees)
1	4430.74	76.73	−45.00
2	2035.38	76.52	−44.49
3	1040.04	76.42	−44.48

closer for Case 6 than for Case 5. As shown in Tables 6 and 7, the miss distances differ resulting from the initial launcher elevation angle of 76.73°, with the miss distance achieved in Case 6 being closer to the target.

7. Conclusions and recommendations

A method has been developed to accurately predict the trajectory of a ballistic missile required to impact a desired position, given only the launch and targeted impact locations. Using missile airframe and performance data, a 6-DOF simulation was developed to generate a trajectory, along which a state transition matrix is propagated. Upon impact, the errors in position due to initial missile orientation are computed using the STM. Using these errors, adjustments to the initial state vector elements corresponding to the initial missile quaternion are calculated. This method is sufficient in converging to a solution within 1500 m tolerance of the target and within three iterations.

A limitation of the simulation is the boost phase modeling. As stated earlier, this is a complex and threat specific portion of any ballistic missile trajectory. To more accurately model this segment of the trajectory, a threat specific guidance computer and autopilot would need to be simulated for each ballistic missile system to be included in the simulation. The most viable option would be to wrap the STM iterative calculation process around an existing ballistic missile 6-DOF model. The iterations would be based upon boost phase parameters, such as booster burn out time and burn out angle instead of launcher angles. This method would not interfere with the iterative, STM-based algorithm which is the focus of this study.

A logical upgrade to the simulation would be the use of an oblate Earth versus the present spherical model being used. This addition will improve simulation accuracy for longer trajectories and trajectories with large polar components. The oblate shape provides more realistic ground tracks and the gravitational effects of oblateness (i.e., $J2$ perturbations) contribute to a more accurate trajectory, especially for long flight times with extended exo-atmospheric portions.

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